

SEMISTABILITY OF CERTAIN BUNDLES ON SECOND SYMMETRIC POWER OF A CURVE

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ABSTRACT. Let C be a smooth irreducible projective curve and E be a stable bundle of rank 2 on C . Then one can associate a rank 4 vector bundle $\mathcal{F}_2(E)$ on $S^2(C)$, the second symmetric power of C . Our goal in this article is to study semistability of this bundle.

1. INTRODUCTION

It has been an interesting and important object to study vector bundles over smooth projective varieties. The moduli space of semistable vector bundles with fixed topological invariants is well understood for the case of curves. However the question of existence of such bundles is open for higher dimensional varieties. In this article we will study the semistability of certain vector bundles on second symmetric power of a smooth projective curve, which arises naturally.

Let C be smooth irreducible projective curve over the fields \mathbb{C} of complex numbers and E be a rank r vector bundle on C . There is a naturally associated vector bundle $\mathcal{F}_2(E)$ of rank $2r$ on the second symmetric power $S^2(C)$ which is defined in Section 2. The stability and semi-stability for case $r = 1$, i.e. when E is a line bundle on C , has been studied and well understood ([3], [2]). In this article we consider the case when rank E is two.

Fixing a point $x \in C$, the image of $\{x\} \times C$ in $S^2(C)$ defines an ample divisor H' on $S^2(C)$, which we denote by $x + C$. We prove the following:

Theorem 1.1. *Let E be a rank two stable vector bundle of even degree $d \geq 2$ on C such that $\mathcal{F}_2(E)$ is globally generated. Then the bundle $\mathcal{F}_2(E)$ on $S^2(C)$ is $\mu_{H'}$ -semistable with respect to the ample class $H' = x + C$.*

Theorem 1.2. *Assume the genus of C greater than 2. Let E be a rank two $(0, 1)$ -stable bundle (defined in Section 4) of odd degree $d \geq 1$ on C such that $\mathcal{F}_2(E)$ is globally generated. Then the bundle $\mathcal{F}_2(E)$ on $S^2(C)$ is $\mu_{H'}$ -semistable with respect to the ample class $H' = x + C$.*

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2. PRELIMINARIES

Let C be a smooth irreducible projective curve over the field of complex numbers \mathbb{C} of genus g . On the space $C \times C$, consider the following involution $C \times C \longrightarrow C \times C, (x, y) \mapsto (y, x)$. The resulting quotient space is denoted by $S^2(C)$, called the second symmetric power of C . It is a smooth irreducible projective surface over \mathbb{C} . Note that, $S^2(C)$ is naturally identified with the set of all degree 2 effective divisors of C . Set

$$\Delta_2 := \{(D, p) \in S^2(C) \times C \mid D = p + q, \text{ for some } q \in C\}.$$

Then Δ_2 is a divisor in $S^2(C) \times C$, called the universal divisor of degree 2. Let q_1 and q_2 be the projections from $S^2(C) \times C$ onto the first and second factors respectively. Then the restriction of the first projection to Δ_2 induces a morphism

$$q : \Delta_2 \longrightarrow S^2(C),$$

which is a two sheeted ramified covering. For any vector bundle E of rank r on C we construct a bundle $\mathcal{F}_2(E) := (q)_*(q_2^*(E) \mid_{\Delta_2})$ of rank $2r$ over $S^2(C)$. From the exact sequence

$$0 \rightarrow \mathcal{O}_{S^2(C) \times C}(-\Delta_2) \rightarrow \mathcal{O}_{S^2(C) \times C} \rightarrow \mathcal{O}_{\Delta_2} \rightarrow 0$$

on $S^2(C) \times C$ we get the following exact sequence on $S^2(C)$

$$0 \rightarrow q_{1*}(q_2^*E \otimes \mathcal{O}_{S^2(C) \times C}(-\Delta_2)) \rightarrow q_{1*}q_2^*E \rightarrow \mathcal{F}_2(E).$$

Define $f : C \times C \rightarrow \Delta_2$ by $(x, y) \mapsto (x + y, x)$. Then f is an identification. Let $p_i : C \times C \rightarrow C$ be the i -th coordinate projection and let $\pi : C \times C \rightarrow S^2(C)$ be the quotient map. Then it's easy to check that $\pi = q \circ f$ and $\mathcal{F}_2(E) = \pi_*p_2^*E$.

Remark 2.1. *Let C be a smooth irreducible projective curve over \mathbb{C} of genus g and let M be a line bundle on C of degree d . Consider the rank two vector bundle $V(M) := \pi_*p_2^*M$ on $S^2(C)$. Using Grothendieck-Riemann-Roch, one can compute the Chern classes of $V(M)$:*

$$c_1(V(M)) = (d - g - 1)x + \theta$$

and

$$c_2(V(M)) = \binom{d - g}{2}x^2 + (d - g)x.\theta + \frac{\theta^2}{2}$$

where x is the image of the cohomology class of $x + C$ in $S^2(C)$, θ is the cohomology class of the pull back of the theta divisor in $\text{Pic}^2(C)$ under the natural map of $S^2(C)$ to $\text{Pic}^2(C)$ [1, Lemma 2.5, Chapter VIII]. Note that the cohomology group $H^4(S^2(C), \mathbb{Z})$ is naturally isomorphic to \mathbb{Z} , and $x^2 = 1, x.\theta = g, \theta^2 = g(g - 1)$.

To find the Chern character of $\mathcal{F}_2(E)$, for any rank r vector bundle E , first choose a filtration of E such that the successive quotients are line bundles and use the fact that $\mathcal{F}_2(\oplus M_k) = \oplus \mathcal{F}_2(M_k)$ where M_k 's are line bundles over C . Then the Chern character of $\mathcal{F}_2(E)$ has the following expression [5]:

$$ch(\mathcal{F}_2(E)) = \text{degree}(E)(1 - \exp(-x)) - r(g - 1) + r(1 + g + \theta)\exp(-x).$$

From the above expression one can easily see that $c_1(\mathcal{F}_2(E)) = (d - r(g + 1))x + r\theta$, where $d = \text{degree } E$.

3. SEMISTABILITY OF $\mathcal{F}_2(E)$, FOR DEGREE E EVEN

Let C be a smooth irreducible projective curve over the field of complex numbers \mathbb{C} of genus g and let E be a rank r vector bundle on C . In this section we will prove the semistability of the vector bundle $\mathcal{F}_2(E)$, when $r = 2$ and degree E is even. We start with the following definitions.

Definition 3.1. Let C be a non-singular irreducible curve. For a vector bundle F on C we define

$$\mu(F) := \frac{\text{degree}(F)}{\text{rank}(F)}.$$

A vector bundle F on C is said to be semistable (respectively, stable) if for every subbundle F' of F we have

$$\mu(F') \leq \mu(F) \text{ (respectively, } \mu(F') < \mu(F)).$$

Definition 3.2. Let X be a smooth irreducible surface and let H be an ample divisor on X . For a coherent torsion free sheaf F on X , we set

$$\mu_H(F) := \frac{\text{degree}_H(F)}{\text{rank}(F)}$$

where $\text{degree}_H(F) = c_1(F) \cdot H$.

A vector bundle F on X is said to be μ_H -semistable (respectively, μ_H -stable), if for every coherent torsion free subsheaf F' of F with $0 < \text{rank}(F') < \text{rank}(F)$, we have

$$\mu_H(F') \leq \mu_H(F) \text{ (respectively, } \mu_H(F') < \mu_H(F)).$$

Theorem 3.3. Let E be a rank two stable vector bundle of even degree $d \geq 2$ on C such that $\mathcal{F}_2(E)$ is globally generated. Then the bundle $\mathcal{F}_2(E)$ on $S^2(C)$ is $\mu_{H'}$ -semistable with respect to the ample class $H' = x + C$.

Remark 3.4. If E is an even degree vector bundle which is a quotient of direct sum of very ample line bundles, i.e. if there is a surjection $\oplus L_i \rightarrow E$ where each L_i is a very ample line bundle on C , then E satisfies the property of Theorem 3.1.

We recall some well known results.

Lemma 3.5. Let $f : X \rightarrow Y$ be a finite surjective morphism of non-singular surfaces, F be a vector bundle on Y , and H be an ample divisor on Y . Assume $f^*(F)$ is $\mu_{f^*(H)}$ -semistable (repectively, $\mu_{f^*(H)}$ -stable). Then F is μ_H -semistable (respectively, μ_H -stable).

Proof: [2, Lemma 4.4].

Lemma 3.6. Let C be a smooth irreducible curve of genus $g \geq 1$ and let K_C be the canonical bundle of C . Let $J^{g-1}(C)$ be the variety of line bundles of degree $g-1$ of C , and let Θ be the divisor on $J^{g-1}(C)$ consisting of line bundles with non-zero sections. Let ξ be a line bundle on C of degree $g-3$ and

$$\nu_\xi : C \times C \rightarrow J^{g-1}(C)$$

be the morphism $(x, y) \mapsto \mathcal{O}_{C \times C}(x + y) \otimes \xi$. Then

$$\nu_\xi^*(\Theta) \cong p_1^*(K_C \otimes \xi^*) \otimes p_2^*(K_C \otimes \xi^*) \otimes \mathcal{O}_{C \times C}(-\Delta)$$

where Δ is the diagonal of $C \times C$ and $p_i : C \times C \rightarrow C$ is the i -th coordinate projection.

Proof: [2, Lemma 4.5].

Using Lemma 3.5, we see that, to prove the semistability of $\mathcal{F}_2(E)$ on $S^2(C)$ with respect to the ample class $x + C$, it is sufficient to prove the semistability of $\pi^*(\mathcal{F}_2(E))$ on $C \times C$ with respect to the ample divisor $H := \pi^*(H') = [x \times C + C \times x]$. By Lemma 3.6, we have $\pi^*(\theta) = (g+1)[x \times C + C \times x] - \Delta$. Since $c_1(\mathcal{F}_2(E)) = (d-2(g+1))x + 2\theta$, we have

$$c_1(\pi^*(\mathcal{F}_2(E))) = d[x \times C + C \times x] - 2\Delta,$$

and

$$\mu_H(\pi^*(\mathcal{F}_2(E))) = \frac{d-2}{2}.$$

First note that the bundle $\pi^*(\mathcal{F}_2(E))$ fits in the following exact sequence on $C \times C$:

$$(1) \quad 0 \rightarrow \pi^*(\mathcal{F}_2(E)) \rightarrow p_1^*(E) \oplus p_2^*(E) \xrightarrow{q} E = p_1^*(E)|_\Delta = p_2^*(E)|_\Delta \rightarrow 0$$

where the map q is given by $q : (u, v) \mapsto u|_\Delta - v|_\Delta$. Let $\phi_i : \pi^*(\mathcal{F}_2(E)) \rightarrow p_i^*(E)$ be the restriction of the projection $p_1^*(E) \oplus p_2^*(E) \rightarrow p_i^*(E)$ to

$\pi^*(\mathcal{F}_2(E)) \subset p_1^*(E) \oplus p_2^*(E)$. Then from the exact sequence (1), we get the following two exact sequences:

$$(2) \quad 0 \rightarrow p_1^*(E) \otimes \mathcal{O}_{C \times C}(-\Delta) \rightarrow \pi^*(\mathcal{F}_2(E)) \xrightarrow{\phi_1} p_2^*(E) \rightarrow 0,$$

and

$$(3) \quad 0 \rightarrow p_2^*(E) \otimes \mathcal{O}_{C \times C}(-\Delta) \rightarrow \pi^*(\mathcal{F}_2(E)) \xrightarrow{\phi_2} p_1^*(E) \rightarrow 0$$

[4, Section 3].

Lemma 3.7. $p_i^*(E)$ is μ_H -stable, $\forall i = 1, 2$.

Proof. Due to symmetry, we will do it only for p_2^*E . Since over a smooth irreducible projective surface double dual of a torsion free sheaf is free, by taking double dual if necessary, we see that to prove stability or semistability it is enough to consider subsheafs which are line bundles. Let L be a line bundle on $C \times C$ which is a subsheaf of p_2^*E such that the quotient, M say, is torsion free. We have an exact sequence

$$0 \longrightarrow L \longrightarrow p_2^*E \longrightarrow M \longrightarrow 0.$$

We restrict this exact sequence to $x \times C$ and $C \times x$, respectively, to obtain the following exact sequences

$$0 \longrightarrow L|_{x \times C} \longrightarrow E \longrightarrow M|_{x \times C} \longrightarrow 0,$$

and

$$0 \longrightarrow L|_{C \times x} \longrightarrow \mathcal{O}_C \oplus \mathcal{O}_C \longrightarrow M|_{C \times x} \longrightarrow 0.$$

From the first exact sequence we get, $\deg(L|_{x \times C}) = c_1(L) \cdot [x \times C] < \mu(E) = \frac{d}{2}$, since E is stable. And from the second exact sequence we get $\deg(L|_{C \times x}) = c_1(L) \cdot [C \times x] \leq 0$. Thus $\deg(L) = c_1(L) \cdot [x \times C + C \times x] < \frac{d}{2} = \mu_H(p_2^*E)$, proving the Lemma. \square

Proof of Theorem 3.3:

Let L be a line bundle which is a subsheaf of $\pi^*(\mathcal{F}_2(E))$ such that the quotient is torsion free. Suppose there is a non-zero homomorphism from L to $p_1^*(E)(-\Delta) := p_1^*(E) \otimes \mathcal{O}_{C \times C}(-\Delta)$. Then $\mu_H(L) < \mu_H(p_1^*(E)(-\Delta)) = \frac{d-4}{2} < \frac{d-2}{2}$. So assume that there is no non-zero map from L to $p_1^*(E)(-\Delta)$. Thus there is an injection $L \rightarrow p_2^*(E)$ so that $\mu_H(L) < \mu_H(p_2^*(E)) = \frac{d}{2}$. Since d is even, $\mu_H(L) \leq \frac{d}{2} - 1 = \frac{d-2}{2}$.

Now let F be a rank two coherent subsheaf of $\pi^*(\mathcal{F}_2(E))$ such that quotient is torsion-free. Then we have the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & p_1^*(E)(-\Delta) & \longrightarrow & \pi^*(\mathcal{F}_2(E)) & \longrightarrow & p_2^*(E) \longrightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & F'' \longrightarrow 0
\end{array}$$

where the vertical arrows are injections. Suppose that both F' and F'' are non-zero. These two are rank 1 coherent sheaf. So we have, $\deg(F') = \mu_H(F') < \mu_H(p_1^*(E)(-\Delta)) = \frac{d-4}{2}$ and $\deg(F'') = \mu_H(F'') < \mu_H(p_2^*(E)) = \frac{d}{2}$. Thus $\mu_H(F) = \frac{1}{2}(\deg(F') + \deg(F'')) < \frac{d-2}{2}$. Now assume at least one of F' and F'' is zero. First let F'' be zero. Then we have an injection $F \rightarrow p_1^*(E)(-\Delta)$ and the cokernel is a torsion sheaf. If the cokernel is supported at only finitely many points, then $\mu_H(F) = \mu_H(p_1^*(E)(-\Delta)) < \frac{d-2}{2}$. If the cokernel is supported at a co-dimension 1 subscheme, then $\mu_H(F) < \mu_H(p_1^*(E)(-\Delta)) < \frac{d-2}{2}$. Now let F' is zero. So we have an injection $F \rightarrow p_2^*(E)$ and the cokernel is a torsion sheaf. If the cokernel is supported at a subscheme of co-dimension 1, then $\mu_H(F) < \mu_H(p_2^*(E)) = \frac{d}{2}$ so that $\mu_H(F) \leq \frac{d-1}{2}$. If $\mu_H(F) = \frac{d-1}{2}$, then the cokernel is supported on a divisor of degree one. Now an effective divisor of degree one on $C \times C$ is of the form $x \times C$ or $C \times x$, for some $x \in C$. Thus $c_1(F)$ is of the form $c_1(p_2^*(E)) + [-x \times C]$ or $c_1(p_2^*(E)) + [-C \times x]$. But $c_1(\pi^*(\mathcal{F}_2(E))) = d[C \times x + x \times C] - 2\Delta$, therefore $c_1((\pi^*(\mathcal{F}_2(E))/F)) = (d+1)[x \times C] - 2\Delta$ or $d[x \times C] + [C \times x] - 2\Delta$. In both the cases the torsion free sheaf $\pi^*(\mathcal{F}_2(E))/F$ restricted to any curve of the form $x \times C$ has negative degree. This gives a contradiction to the fact that $\pi^*(\mathcal{F}_2(E))$ is generated by sections. Thus we have, $\mu_H(F) \leq \frac{d-2}{2}$.

If the cokernel is supported only at finitely many points then $\mu_H(F) = \mu_H(p_2^*(E)) = \frac{d}{2}$. In this case, F is a rank two stable sheaf and hence it is isomorphic to $p_2^*(E)$. So the exact sequence (2) splits, i.e., $\pi^*(\mathcal{F}_2(E)) \cong p_1^*(E)(-\Delta) \oplus p_2^*(E)$. Since $p_1^*(E)|_{x \times C}$ is trivial, $\deg(p_1^*(E)(-\Delta)|_{x \times C}) < 0$. This contradicts the fact that $\mathcal{F}_2(E)$ and hence $\pi^*(\mathcal{F}_2(E))$ is globally generated.

Let F be a rank 3 coherent subsheaf of $\pi^*(\mathcal{F}_2(E))$ such that the quotient is torsion free. Then we have the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & p_1^*(E)(-\Delta) & \longrightarrow & \pi^*(\mathcal{F}_2(E)) & \longrightarrow & p_2^*(E) \longrightarrow 0 \\
& & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & F'' \longrightarrow 0
\end{array}$$

where the vertical arrows are injections. We have two possibilities: (I) $\text{rank} F' = 2$ and $\text{rank} F'' = 1$; (II) $\text{rank} F' = 1$ and $\text{rank} F'' = 2$. Suppose that $\text{rank} F' = 2$ and $\text{rank} F'' = 1$. By the arguments above,

we have, $\mu_H(F') \leq \frac{d-4}{2}$ and $\mu_H(F'') < \frac{d}{2}$. So

$$\mu_H(F) < \frac{3d-8}{6} < \frac{d-2}{2}.$$

Now assume that $\text{rank} F' = 1$ and $\text{rank} F'' = 2$. In this case, we have, $\mu_H(F') < \frac{d-4}{2}$ and $\mu_H(F'') \leq \frac{d}{2}$. If d is even, $\mu_H(F') \leq \frac{d-4}{2} - 1$, hence $\mu_H(F) \leq \frac{3d-6}{6} = \frac{d-2}{2}$.

4. SEMISTABILITY OF $\pi^*(\mathcal{F}_2(E))$ FOR DEGREE E ODD

In this section we will prove that the semi-stability of $\pi^*(\mathcal{F}_2(E))$ when degree E is odd. First let's recall some definitions.

Definition 4.1. Let E be a non-zero vector bundle on C and $k \in \mathbb{Z}$, we denote by $\mu_k(E)$ the rational number

$$\mu_k(E) := \frac{\text{degree}(E) + k}{\text{rank}(E)}.$$

We say that the vector bundle E is (k, l) -stable (resp. (k, l) -semistable) if, for every proper subbundle F of E we have

$$\mu_k(F) < \mu_{-l}(E/F) \text{ (resp. } \mu_k(F) \leq \mu_{-l}(E/F) \text{)}.$$

Note that usual Mumford stability is equivalent to $(0, 0)$ -stability. If $g \geq 3$, then there always exists a $(0, 1)$ -stable bundle and if $g \geq 4$, then the set of $(0, 1)$ -stable bundles form a dense open subset of the moduli space of stable bundles over C of rank 2 and degree d . [6, Section 5]

Theorem 4.2. Assume the genus of C greater than 2. Let E be a rank two $(0, 1)$ -stable bundle of odd degree $d \geq 1$ on C such that $\mathcal{F}_2(E)$ is globally generated. Then the bundle $\mathcal{F}_2(E)$ on $S^2(C)$ is $\mu_{H'}$ -semistable with respect to the ample class $H' = x + C$.

Proof. Let L be a line bundle which is a subsheaf of $\pi^*(\mathcal{F}_2(E))$ such that the quotient is torsion free. Suppose there is a non-zero homomorphism from L to $p_1^*(E)(-\Delta) := p_1^*(E) \otimes \mathcal{O}_{C \times C}(-\Delta)$. Then $\mu_H(L) < \mu_H(p_1^*(E)(-\Delta)) = \frac{d-4}{2} < \frac{d-2}{2}$. So assume that there is no non-zero map from L to $p_1^*(E)(-\Delta)$. Thus there is an injection $L \rightarrow p_2^*(E)$. Now consider the exact sequence,

$$(4) \quad 0 \longrightarrow L \longrightarrow p_2^*(E) \longrightarrow M \longrightarrow 0,$$

where M is a sheaf of rank 1. Restricting the above exact sequence to $C \times x$, we see that, $c_1(L).[C \times x] \leq 0$. On the other hand, restricting the above exact sequence to $C \times x$ and using that E is $(0, 1)$ -stable, we get that $c_1(L).[C \times x] < \frac{d-1}{2}$. Since L is a line bundle, $c_1(L).[C \times x] \leq \frac{d-3}{2}$. So we have $\mu_H(L) \leq \frac{d-3}{2} < \frac{d-2}{2}$.

Let's assume F be a rank two coherent subsheaf of $\pi^*(\mathcal{F}_2(E))$ such that quotient is torsion-free. Then we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & p_1^*(E)(-\Delta) & \rightarrow & \pi^*(\mathcal{F}_2(E)) & \longrightarrow & p_2^*(E) \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & F'' \longrightarrow 0 \end{array}$$

where the vertical arrows are injections. We need to consider three different cases: (I) $\text{rank } F' = 1 = \text{rank } F''$; (II) $F'' = 0$; (III) $F' = 0$. In each of these cases, we can argue exactly as in the case of even degree to conclude that $\mu_H(F) \leq \frac{d-2}{2} = \mu_H(\pi^*\mathcal{F}_2(E))$.

Now assume F is subsheaf of $\pi^*\mathcal{F}_2(E)$ rank 3. Then again we have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & p_1^*(E)(-\Delta) & \rightarrow & \pi^*(\mathcal{F}_2(E)) & \longrightarrow & p_2^*(E) \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & F'' \longrightarrow 0 \end{array}$$

where the vertical arrows are injections. We have two possibilities: (I) $\text{rank } F' = 2$ and $\text{rank } F'' = 1$; (II) $\text{rank } F' = 1$ and $\text{rank } F'' = 2$. Using the same argument as in Theorem 3.3, we can show that in the case of (I), $\mu_H(F) < \frac{d-2}{2}$. Now consider the case (II). In this case, restricting the exact sequence $0 \rightarrow F' \rightarrow p_1^*(E)(-\Delta)$ to $x \times C$ and $C \times x$, we get that

$$0 \rightarrow F'|_{x \times C} \rightarrow \mathcal{O}_C(-x)$$

and

$$0 \rightarrow F'|_{C \times x} \rightarrow E \otimes \mathcal{O}_C(-x).$$

From these two exact sequences and using the fact that E is $(0, 1)$ -stable we see that $\mu_H(F') < \frac{d-4}{2}$ and hence $\mu_H(F') \leq \frac{d-6}{2}$. Also using the same argument as above, we have, in any case, $\mu_H(F'') \leq \frac{d}{2}$. Combining all these, we get that $\mu_H(F) < \frac{d-2}{2}$. □

5. RESTRICTION TO CURVES OF THE FORM $x + C$

In this section we will investigate the restriction of $\mathcal{F}_2(E)$ to the curves of the form $x + C$ where $x + C$ is the reduced divisor of $S^2(C)$ whose support equals to $\{x + c : c \in C\}$. For this we have the following theorem.

Theorem 5.1. *Let C be a smooth irreducible projective curve over \mathbb{C} of genus g and let E be a rank 2 vector bundle on C of degree $d \geq 3$. Then for any $x \in C$, $\mathcal{F}_2(E)|_{x+C}$ is not semistable.*

Proof. First note that, since E is locally free, p_2^*E is flat over $S^2(C)$ and using the base change formula we get

$$\mathcal{F}_2(E)|_{x+C} = \pi_*(p_2^*E|_{\pi^{-1}(x+C)}).$$

Also we have the following exact sequence

$$0 \rightarrow p_2^*E|_{\pi^{-1}(x+C)} \rightarrow p_2^*E|_{x \times C} \oplus p_2^*E|_{C \times x} \rightarrow E|_{(x,x)} \rightarrow 0.$$

From this exact sequence and using the fact that $\pi|_{x \times C} : x \times C \rightarrow x + C$ and $\pi|_{C \times x} : C \times x \rightarrow x + C$ are isomorphisms and $p_2^*E|_{x \times C} = E$ and $p_2^*E|_{C \times x} = E_x \otimes \mathcal{O}_C$, we get an injective map

$$0 \rightarrow E \otimes \mathcal{O}_C(-x) \rightarrow \mathcal{F}_2(E)|_{x+C}.$$

Now the degree of $E \otimes \mathcal{O}_C(-x) = d-2$ and that of $\mathcal{F}_2(E)|_{x+C} = d-2$. So the cokernel is rank 2 coherent sheaf of degree zero. If it is torsion free then clearly $\mathcal{F}_2(E)|_{x+C}$ is not semistable. If the cokernel has torsion, then there is an effective divisor D such that the above map factors through $E \otimes \mathcal{O}_C(-x) \otimes \mathcal{O}_C(D)$ and in this case the cokernel will be again torsion free. But in this case the degree of the cokernel will be of negative degree. So in this case $\mathcal{F}_2(E)|_{x+C}$ has a torsion free quotient of negative degree. Hence it is not semistable. \square

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